Mutual Chern-Simons Landau-Ginzburg theory for continuous quantum phase transition of Z_2 topological order

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In this paper, we develop an effective mutual Chern-Simons Landau-Ginzburg (MCSLG) theory to describe the continuous topological quantum phase transition (TQPT). In particular, we consider the TQPT between a spin-polarized phase (a state without topological order) and a Z_2 topologically ordered state. The TQPT is not induced by spontaneous symmetry breaking. Instead the Z_2 topological order is broken down by the condensation of Z_2 charged quasiparticles. By generalizing the hierarchy theory of fractional quantum Hall effect to Z_2 topological order, we show that the TQPT belongs to the universal class of three-dimensional Ising phase transition. In the end, we applied the MCSLG theory to the toric code model.

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I. INTRODUCTION

Landau developed a systematic theory for phases in classical statistical systems. Different orders are characterized by different symmetries. The phase transition between ordered phase and disordered phase is accompanied with symmetry breaking. To describe continuous phase transition, the local order parameter is introduced which changes from a finite value to zero across the critical point of phase transition. However, Landau theory cannot describe all the continuous phase transition such as the transitions between topological ordered phases.[1] This kind of phase transitions, known as topological quantum phase transition (TQPT), has attracted considerable attention in recent years.[1–15]

Since those quantum-phase transitions are transitions between two quantum states with the *same* symmetry, they are fundamentally different from the usual continuous transitions, which change the symmetry between the states. First, the TQPT of the toric-code model in a magnetic field was studied in Ref. 7. The numerical simulations demonstrate the condensation of “magnetic” excitations and the confinement of “electric” charges of the phase transition out of the topological phase. And the TQPT belongs to the universal class of the three-dimensional (3D) Ising model. In Ref. 12 it is revealed that the closed string operators can be order parameters for the TQPT of the transverse Wen-plaquette model by mapping it onto that in one-dimensional (1D) transverse Ising model. In particular, the duality relationship between open string and closed string is explored. Recently, a global phase diagram of the toric-code model in a magnetic field is obtained.[14]

We know that the Ginzburg-Landau theory provided us a general frame work to described all kinds of symmetry breaking phase transitions. However, the Ginzburg-Landau theory does not apply to continuous phase transitions that changes topological orders. Therefore, it is very desirable to develop a general frame work continuous topological phase transitions such as Ginzburg-Landau theory for symmetry breaking transitions. In this paper, we use mutual Chern-Simons theory to describe topological phases. Mutual Chern-Simons theory can describe a large class of time-reversal and parity symmetric topological phases. Using the mutual Chern-Simons theory, we find that the continuous topological phase transitions can induced by the condensation of particles that carry the charge of the gauge fields in the Chern-Simons theory. Thus, a general frame work for continuous topological phase transition can be formulated using mutual Chern-Simons Landau-Ginzburg (MCSLG) theory. From the effective MCSLG theory, we find the universal class of the TQPT between a Z_2 topological order and a spin-polarized phase without topological order.

The remainder of the paper is organized as follows. In Sec. II, we introduce the $U(1)\times U(1)$ mutual Chern-Simons theory for Z_2 topological order. In Sec. III, we consider the continuous phase transition from the Z_2 topologically ordered phase to another phase induced by the condensation of U(1) charged particles. We generalize the fractional quantum Hall (FQH) hierarchy theory to study the quantum phase transition, in particular to learn the nature of the charged-particle condensed phase after the transition. We find that the charged-particle condensed phase has a trivial topological order. The charged-particle condensation describes the continuous phase transition between Z_2 topologically ordered state and spin-polarized state. Then in Sec. IV, we use the toric-code model as an example to show the properties of the quantum phase transition out of Z_2 topological order. Finally, the conclusions are presented in Sec. V.

II. $U(1)\times U(1)$ MUTUAL CHERN-SIMONS THEORY FOR Z_2 TOPOLOGICAL ORDER

Z_2 topological order is the simplest topological order.[1] The low-energy effective theory for those Z_2 topologically ordered states is a Z_2 gauge theory with three types of quasiparticles: Z_2 charge, Z_2 vortex, and fermions.[16] Z_2 charge and Z_2 vortex are all bosonic excitations with mutual π statistics between them. The fermions can be regarded as bound states of a Z_2 charge and a Z_2 vortex.

It is pointed out that the continuum effective theory of two dimensional Z_2 gauge theory is a $U(1)\times U(1)$ mutual Chern-Simons (MCS) theory. In a $U(1)\times U(1)$ MCS theory, there are two types of *assistant* gauge fields A_μ and a_μ, which Z_2 vortex and Z_2 charge couple, respectively. A mutual Chern-Simons term is introduced between A_μ and a_μ,

$$\mathcal{L}_{\mathrm{MCS}} = \frac{1}{\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda. \tag{1}$$

Then the effective Lagrangian is obtained as

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4e_A^2}(\partial_\nu A_\mu)^2 - \frac{1}{4e_a^2}(\partial_\nu a_\mu)^2 + \mathcal{L}_{\mathrm{MCS}}. \tag{2}$$

Here, e_A and e_a are coupling constants of gauge fields A_μ and a_μ, respectively.

Including the quasiparticles, the effective theory of it becomes

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4e_a^2}(\partial_\nu a_\mu)^2 - \frac{1}{4e_A^2}(\partial_\nu A_\mu)^2 + \mathcal{L}_{\mathrm{MCS}} + \mathcal{L}_v + \mathcal{L}_c. \tag{3}$$

$\mathcal{L}_v$ and $\mathcal{L}_c$ are the Lagrangians of Z_2 vortex and Z_2 charge. In general $\mathcal{L}_v$ and $\mathcal{L}_c$ are

$$\mathcal{L}_v = |(\partial_\mu - iA_\mu)\phi_1|^2 + m_1^2\phi_1^*\phi_1 + \frac{\tilde{m}_1^2}{2}(\phi_1^2 + \mathrm{H.c.}) + g_1|\phi_1^*\phi_1|^2,$$

$$\mathcal{L}_c = |(\partial_\mu - ia_\mu)\phi_2|^2 + m_2^2\phi_2^*\phi_2 + \frac{\tilde{m}_2^2}{2}(\phi_2^2 + \mathrm{H.c.}) + g_2|\phi_2^*\phi_2|^2. \tag{4}$$

Here, the complex fields $\phi_\alpha(\alpha=1,2)$ are bosonic fields denoting Z_2 vortex and Z_2 charge. Since $\phi_{1,2}$ is conserved only mod 2 for a Z_2 topological state on lattice, there may exist the terms $\frac{\tilde{m}_1^2}{2}(\phi_1^2+\mathrm{H.c.})$ and $\frac{\tilde{m}_2^2}{2}(\phi_2^2+\mathrm{H.c.})$, with which the ground state has a Z_2 type of local symmetry and is compatible to its lattice realization.

All the topological properties, including topological degeneracy, quantum numbers, and edge states, agree, indicating the equivalence between the Z_2 topological states on lattice and the $\mathrm{U}(1)\times\mathrm{U}(1)$ MCS theory.[17] The MCS theory in above equations has been used to study the topological order in frustrated Josephson junction arrays.[18,19] In addition, similar MCS theory was proposed to be the effective gauge theory of doped Mott insulator[20,21] for high T_c superconductors.

III. HIERARCHY THEORY OF THE QUANTUM PHASE TRANSITION

In the following, we will use the MCSLG theory to develop a unified theory of TQPTs. First, in the MCSLG theory, the phase with $m_1^2-\tilde{m}_1^2>0$ and $m_2^2-\tilde{m}_2^2>0$ is the topologically ordered phase with Z_2 topological order. We would like to show that the spin-polarized phase without topological order is characterized by the MCS theory with quasiparticle condensations (say with $m_2^2-\tilde{m}_2^2<0$). On one hand, it seems obvious that when ϕ_2 condenses ($\phi_2\neq 0$), the MCSLG should describe a topologically trivial phase. On the other hand, one worries that the vortex of the ϕ_2 field may carry nontrivial statistics, and in that case the ϕ_2 condensed phase would have a nontrivial topological order.

To combine the two different quantum states into one picture, we generalize the hierarchy theory of fractional quantum hall states.

In the following, we will use the approach developed for fractional quantum Hall states to actually calculate the topological order of the ϕ_2 condensed phase.[22] Let us give a short introduction to the FQH hierarchy theory. Ten years ago, it is found that all fractional Abelian quantum Hall fluids can be classified by the K matrices.[23] When a gas of quasiparticles of Abelian quantum Hall state labeled by $\check{l}$ on top of the (K,q) state are condensed, quantum-phase transition occurs and one may get another type of Abelian quantum hall state labeled by

$$K' = \begin{pmatrix} K & l \\ -l^T & N \end{pmatrix} \tag{5}$$

and

$$q' = \begin{pmatrix} q \\ 0 \end{pmatrix}. \tag{6}$$

Here, K is a symmetric integer matrix and q an integer vector, N an even integer. Then one can use the hierarchy theory to study the quantum phase transition between different topological orders in Abelian quantum hall state.[2,4]

In particular, we can also use the K-matrices representation to denote Z_2 topological order. The MCS term in effective Lagrangian (2) is naturally written into

$$\mathcal{L}_{\mathrm{MCS}} = \sum_{I,J}\frac{K_{IJ}}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu^I\partial_\nu a_\lambda^J \tag{7}$$

where $K=\begin{pmatrix} 0 & 2 \\ 2 & 0\end{pmatrix}$ and $a_\mu^{I=1}=A_\mu$, $a_\mu^{I=2}=a_\mu$. The quasiparticles are labeled by integer vectors $l=\begin{pmatrix} q_v \\ q_c\end{pmatrix}$. Such a quasiparticle will carry the following flux

$$\begin{pmatrix} \int d^2x\epsilon^{ij}\partial_i a_j^1 \\ \int d^2x\epsilon^{ij}\partial_i a_j^2 \end{pmatrix} = \pi\begin{pmatrix} q_c \\ q_v \end{pmatrix}. \tag{8}$$

In addition, for Z_2 topological order, Z_2 vortex and Z_2 charge have only mod 2 conservation. This means that the flux of a_μ^I is not conserved. They can change by multiples of 2π.

To calculate the topological order in the ϕ_2 condensed phase, we will use a dual description where we introduce a U(1) gauge field b_μ to describe the density j^0 and current j^i of the ϕ_2 field: $j^\mu=\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$. The mod 2 conservation of ϕ_2 implies that the flux of b_μ can change by 4π. The charge of b_μ is quantized as integers which represents a vortex in the ϕ_2 field. Now the effective Lagrangian for the ϕ_2 condensed phase turns into

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4e_a^2}(f_{\mu\nu})^2 - \frac{1}{4e_A^2}(F_{\mu\nu})^2 - \frac{1}{4e_b^2}(B_{\mu\nu})^2 + \frac{1}{\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu a_\lambda + \mathcal{L}_v + \mathcal{L}_c, \tag{9}$$

where

$$\mathcal{L}_c = |(\partial_\mu - ia_\mu - ib_\mu)\phi_2^v|^2 + m_2^2(\phi_2^v)^*\phi_2^v + g_2|(\phi_2^v)^*\phi_2^v|^2 \tag{10}$$

where ϕ_2^v is the field that describes the vortices of ϕ_2. The MCS terms become

$$\sum_{I,J} \frac{\tilde{K}_{IJ}}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu^I\partial_\nu a_\lambda^J \tag{11}$$

With $a_\mu^{I=1}=A_\mu$, $a_\mu^{I=2}=a_\mu$, $a_\mu^{I=3}=b_\mu$, and

$$\tilde{K} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{12}$$

Therefore, Z_2 topologically ordered state described by $K = \left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$ will transform into another quantum state described by another K matrix $\tilde{K}$ via a TQPT. Then continuous quantum phase transition between the two quantum states $\tilde{K}$ and K can happen. The TQPT at $m_2^2 - \tilde{m}_2^2 = 0$ is the transition from the $\langle\phi_2\rangle = 0$ phase (the K phase) to the $\langle\phi_2\rangle \neq 0$ phase (the $\tilde{K}$ phase).

Does the $\tilde{K}$ phase have a trivial topological order? We note that after a $SL(3,Z)$ transformation,

$$W = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \end{pmatrix}, \tag{13}$$

which redefine that gauge fields $(A_\mu, a_\mu, b_\mu) = (A'_\mu, a'_\mu, b'_\mu)W$, $\tilde{K}$ is transformed into

$$K' = W\tilde{K}W^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

K' describes the same topological order as $\tilde{K}$. What is the topological order described by K'? According to Haldane's topological instability,[24,25] we can drop the two by two block $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The topological order described by K' is the same as that described by the one by one K matrix $K=(0)$. Thus the K' state (the ϕ_2 condensed state) has a trivial topological order.

However, $K=(0)$ appears to imply that the b'_μ gauge field is massless. In fact b'_μ will obtain a mass gap from an instanton effect.[26] We note that a 2π change in the flux of b'_μ correspond to a 2π change in the flux of a_μ^1 and a -4π change in the flux of b_μ. Thus the flux of b'_μ is not conserved. It can change by 2π. Thus, instantons represented by the space-time monopoles are allowed. According to Ref. 26, the instanton effect will gap he b'_μ field and put the b'_μ field into a confined phase. Such a confined phase corresponds to the spin-polarized phase.

The ϕ_2 condensation corresponds to a transition from the Z_2 topologically ordered state to the spin-polarized phase. We can use the MCSLG theory to study such a TQPT. At the critical point of $m_2^2 - \tilde{m}_2^2 = 0$, all gauge fields have mass gaps and only mediate short range interactions due to the Chern-Simons term, and hence are irrelevant to the low energy physics. This suggest the following effective theory at the critical point

$$\mathcal{L}_{\text{eff}} = |\partial_\mu\phi_2|^2 + m_2^2\phi_2^*\phi_2 + \frac{\tilde{m}_2^2}{2}(\phi_2^2 + \text{H.c.}) + g_2|\phi_2^*\phi_2|^2. \tag{15}$$

This $2+1$D φ^4 model belonging to the universal class of the finite-temperature 3D Ising phase transition.

IV. APPLICATION TO THE TORIC-CODE MODEL

In the end, we give an example of the TQPT in two dimensional spin models. In the last decade, several exact soluble spin models with different topological orders were found, such as the toric-code model,[6] the Wen-plaquette model[5] and the Kitaev model on a hexagonal lattice.[27] Here, we will focus on the toric-code model described by the Hamiltonian[6]

$$H = -A\sum_{i \in \text{even}} Z_i - B\sum_{i \in \text{odd}} X_i. \tag{16}$$

where

$$Z_i = \sigma_i^z\sigma_{i+\hat{e}_x}^z\sigma_{i+\hat{e}_x+\hat{e}_y}^z\sigma_{i+\hat{e}_y}^z, \quad X_i = \sigma_i^x\sigma_{i+\hat{e}_x}^x\sigma_{i+\hat{e}_x+\hat{e}_y}^x\sigma_{i+\hat{e}_y}^x$$

with $A>0$, $B>0$. $\sigma_i^{x,y,z}$ are Pauli matrices on sites, i. The ground state denoted by $Z_i \equiv +1$ and $X_i \equiv +1$ at each site is a Z_2 topological state. In this model Z_2 vortex is defined as $Z_i = -1$ at even subplaquette and Z_2 charge is $X_i = -1$ at odd subplaquette. These quasiparticles in such exactly solved model have flat band—the energy spectrums are $E_v = 2A$ and $E_c = 2B$ for Z_2 vortex and Z_2 charge, respectively. In other words, the quasiparticles cannot move at all.

Under a perturbation

$$H' = h^z\sum_i \sigma_i^z, \tag{17}$$

Z_2 charges begin to hop along diagonal directions $(\hat{e}_x \pm \hat{e}_y)$ while Z_2 vortex also cannot move. As shown in Fig. 1, for a Z_2 charge living at i plaquette $X_i = -1$, when σ_i^z acts on $i+\hat{e}_x$ site, it hops to $i+\hat{e}_x-\hat{e}_y$ plaquette denoted by $X_{i+\hat{e}_x-\hat{e}_y} = -1$. Moreover, a pair of Z_2 charges at i and $i+\hat{e}_x-\hat{e}_y$ plaquettes can be created by the operation of σ_i^z,

$$X_i = +1 \to X_i = -1, \quad X_{i+\hat{e}_x-\hat{e}_y} = +1 \to X_{i+\hat{e}_x-\hat{e}_y} = -1 \tag{18}$$

or annihilated also

$$X_i = -1 \to X_i = +1, \quad X_{i+\hat{e}_x-\hat{e}_y} = -1 \to X_{i+\hat{e}_x-\hat{e}_y} = +1.$$

In the following part we use the perturbative method in Refs. 15 and 28–31 to describe the effective Hamiltonian of quasiparticles. In the perturbative method, spin operators are represented by hopping terms of quasiparticles,

$$\sigma_i^x \to (\phi_{1,i}^\dagger\phi_{1,i+e_x\pm e_y} + \phi_{1,i}^\dagger\phi_{1,i+e_x\pm e_y}^\dagger + \text{H.c.}),$$

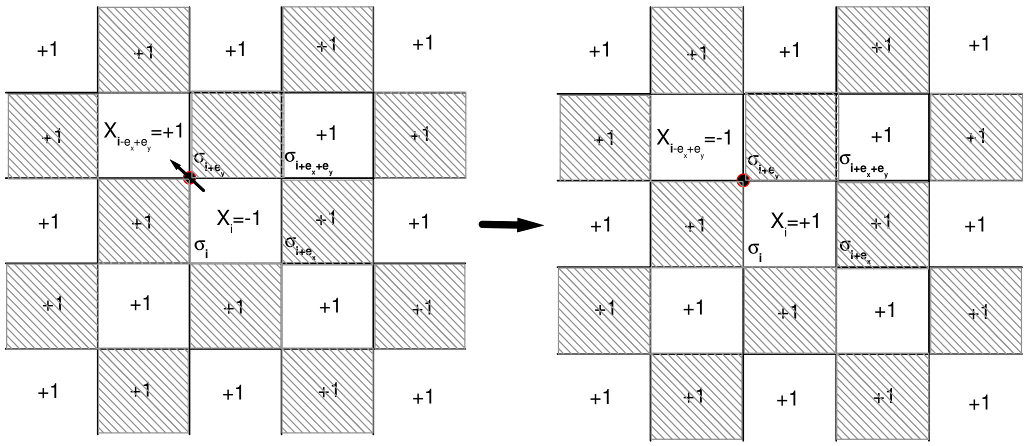

FIG. 1. (Color online) Hopping of the Z_2 charge from i plaquette to $i+\hat{e}_x-\hat{e}_y$ plaquette. The white plaquettes denote odd subplaquettes.

$$\sigma_i^z \to (\phi_{2,i}^\dagger \phi_{2,i+e_x\pm e_y} + \phi_{2,i}^\dagger \phi_{2,i+e_x\pm e_y}^\dagger + \text{H.c.}). \tag{19}$$

Here, $\phi_{\alpha,i}^\dagger$ is the generation operator of the quasiparticles. $\alpha=1,2$ represent Z_2 vortex, Z_2 charge in the toric-code model, respectively. $\langle i,j\rangle$ denote two nearest positions. In addition, one should add a single occupation constraint (hard-core constraint) as

$$(\phi_{\alpha,i}^\dagger)^2|\Psi\rangle = 0 \ \text{ or } \ \phi_{\alpha,i}^\dagger|\Psi\rangle = \phi_{\alpha,i}|\Psi\rangle \tag{20}$$

where $|\Psi\rangle$ denotes quantum state of the toric-code model.

Then, the perturbative effective Hamiltonian of Z_2 charge is given as

$$H_c = 2B\sum_{i\in \text{odd}} \phi_{2,i}^\dagger \phi_{2,i} + h_z \sum_{\langle i,j\rangle \in \text{odd}} (\phi_{2,i}^\dagger \phi_{2,j} + \phi_{2,i}^\dagger \phi_{2,j}^\dagger) + \text{H.c.} \tag{21}$$

In the low energy limit, the mass gaps of (Fig. 2) Z_2 charge are obtained as

$$m_2 = 2B - 4h_z, \ \ \tilde{m}_2 = 4h_z. \tag{22}$$

On the other hand, because a Z_2 vortex cannot move under such external field, the effective Hamiltonian of Z_2 vortex is trivial,

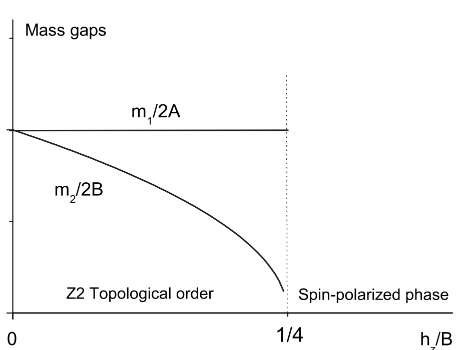

FIG. 2. The mass gaps of quasiparticles.

$$H_v = 2A\sum_{i\in \text{even}} \phi_{1,i}^\dagger \phi_{1,i}. \tag{23}$$

The mass gap of Z_2 vortex is always $m_1=2A$. Then the critical point is defined by zero energy mass gap of Z_2 charge as

$$m_2^2 - \tilde{m}_2^2 = 4B^2\left(1-\frac{4h_z}{B}\right) = 0,$$

where $h_z=\frac{B}{4}$. Near TQPT, the spin correlated function of $\sigma_i^z\sigma_j^z$ has a scaling law as

$$\langle \sigma_i^z \sigma_j^z\rangle \to \frac{1}{|i-j|}, \ \ |i-j| \to \infty. \tag{24}$$

While the spin correlated functions $\sigma_i^x\sigma_j^x$ and $\sigma_i^y\sigma_j^y$ decay exponentially.

The TQPT in toric-code model between a spin-polarized phase and a topological phase can be described by the MCSLG theory. From the results of MCSLG theory, such TQPT belongs to the universal class of 3D Ising phase transition. In Ref. 7, it is found that in the limit of a large Z_2 charge gap, i.e., $A \gg B > 0$, the toric-code model is equivalent to the Z_2 lattice gauge model which is dual to the transverse field Ising model (with both nearest and next-nearest neighbor Ising interactions), and can be mapped to a classical 3D Ising model.[32] Therefore, the results from different approaches match each other.

V. CONCLUSION

To summarize, in this paper, we develop an effective theory–*the mutual Chern-Simons Landau-Ginzburg theory* to describe continuous quantum phase transition between a topologically ordered state and a nontopologically ordered state. Z_2 topological order is described by $U(1)\times U(1)$ MCS effective theory that is characterized by K matrix; while on the other side of the TQPT, it is a trivial quantum insulator. The continuous TQPT is not induced by spontaneous symmetry breaking; instead it is broken down by the condensation of quasiparticles. Using the FQH hierarchy theory,[22] we show that the TQPT belongs to the universal class of 3D Ising phase transition. In the end, we applied the MCSLG theory to the toric-code model.

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